1. Let $f : [1, \infty) \to [0, \infty)$ be any continuous decreasing function. Then show that

$$\sum_{1}^{\infty} f(j) \text{is summable} \Leftrightarrow \int_{1}^{\infty} f(u)du < \infty.$$

[3]

Solution: We will show

$$\int_{1}^{\infty} f(x)\,dx \leq \sum_{n=1}^{\infty} f(n) \leq f(1) + \int_{1}^{\infty} f(x)\,dx \tag{1}$$

Note that showing (1) is enough for the solution. Since f is a monotone decreasing function, $f(x) \leq f(n), \forall x \in [n, \infty)$ and $f(n) \leq f(x), \forall x \in [1, n]$. Hence for every integre $n \geq 1$,

$$\int_{n}^{n+1} f(x)\,dx \leq \int_{n}^{n+1} f(n)\,dx = f(n) \tag{2}$$

and, for every integer $n \geq 2$,

$$f(n) = \int_{n-1}^{n} f(n)\,dx \leq \int_{n-1}^{n} f(x)dx. \tag{3}$$

By summation over all n from 1 to some larger integer M, we get from (2)

$$\int_{1}^{M+1} f(x)\,dx \leq \sum_{n=1}^{M} f(n)$$

and from (3)

$$\sum_{n=1}^{M} f(n) \leq f(1) + \int_{1}^{M} f(x)\,dx.$$

Combining these two estimates yields

$$\int_{1}^{M+1} f(x)\,dx \leq \sum_{n=1}^{M} f(n) \leq f(1) + \int_{1}^{M} f(x)\,dx.$$

Letting M tend to infinity, the bounds in (1) follows. $\square$

2. Let $a_1, a_2, \ldots$ be any sequence of complex numbers. If $\sum_1^\infty |a_n|$ is summable, then $\sum_1^\infty a_n$ is summable. [2]

Solution: Let n and m be any two positive integers such that $n \leq m$ then $\left|\sum_{j=n}^m a_j\right| \leq \sum_{j=n}^m |a_j|$. Thus $\sum_{j=1}^\infty a_j$ is cauchy whenever $\sum_{j=1}^\infty |a_j|$ is so. □

3. Let $\{x_1, x_2, \ldots\}$ be a sequence of nonzero complex numbers. Assume that

$$\limsup_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = L < 1$$

Then $\sum_1^\infty |x_n| < \infty$ [3]

Solution: Theorem 3.34 in *Principles of Mathematical Analysis* by *Walter Rudin* □

4. The series $\sum_1^\infty a_n b_n$ is summable if $A_n = a_1 + a_2 + \ldots + a_n$ is a bounded sequence and the sequence b_n decreases to 0. [4]

Solution: Theorem 3.42 in *Principles of Mathematical Analysis* by *Walter Rudin* □

5. Let w_1, w_2, w_3 be the roots of $x^3 - 1 = 0$. Define $a_n = w_3$ if n is divisible 3, $a_n = w_2$ if $n \equiv 2 \mod 3$ and $a_n = w_1$ if $n \equiv 1 \mod 3$. Show that $\sum_1^\infty \frac{a_n}{\log(n+100)}$ is summable. [3]

Solution: We will apply the result in Question 4. Note that $\omega_1 + \omega_2 + \omega_3 = 0$, therefore, $\{a_n\}$ satisfies the hypothesis of Question 4. Also monotonic decreasing property in Question 4 is satisfied by $b_n = \frac{1}{\log(n+100)}$. Hence the given series is convergent. □

6. Let $a_1, a_2, a_3, \ldots$ be a real sequence bounded below. Let $\alpha = \liminf_{j\to\infty} a_j$. Then for each $\delta > 0$, show that there exists k_0 such that $a_k \geq \alpha - \delta$ for all $k \geq k_0$. [1]

Solution: Note that α is the infimum of all the subsequential limits of the sequence $\{a_n\}$. Therefore no subsequence can converge to a number less than α. Hence for any $\delta > 0$ there exists a $k_0 \in \mathbb{N}$ such that $a_k > \alpha - \delta$, $\forall k \geq k_0$. □

Discuss the summability of the following examples

7. $\sum_{n=1}^\infty p^n n^p \qquad p > 0$ [3]

Solution: Apply ratio test. We get $\limsup_{n\to\infty} \frac{p^{n+1}(n+1)^p}{p^n n^p} = p$. So the series converges for $0 < p < 1$. If $p \geq 1$ then the series can't converge as the $n-$th term doesnt tend to zero. □

8. $\sum\limits_{100}^{\infty} \frac{1}{n \log n (\log\log n)^p}$ $\qquad p > 0$ [3]

Solution: We will use the result in Question 1. *Case 1:* $p = 1$.

$\frac{d}{dx} \log\log\log x = \frac{1}{x \log x (\log\log x)}$. Therefore, $\int\limits_{100}^{\infty} \frac{dx}{x \log x (\log\log x)} = \log\log\log x|_{100}^{\infty} = \infty$. Hence the given series diverges in this case.

Case 2: $0 < p < 1$.

Here we have $\frac{1}{n \log n (\log\log n)^p} > \frac{1}{n \log n (\log\log n)}$. Hence by using *Case 1* and comparison test we get the series diverges in this case also.

Case 3: $p > 1$.

Let $p = 1 + \epsilon$. We have $-\frac{d}{dx} \frac{1}{\epsilon (\log\log x)^\epsilon} = \frac{1}{x \log x (\log\log x)^{1+\epsilon}}$.

Hence $\int\limits_{100}^{\infty} \frac{1}{x \log x (\log\log x)^{1+\epsilon}} = -\frac{1}{\epsilon (\log\log x)^\epsilon}|_{100}^{\infty} < \infty$ and the series converges by Question 1.

□

9. $\sum\limits_{100}^{\infty} n^p (\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}})$ [3]

Solution: *Case 1:* $p < 0$.

Taking $a_n = \frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}$ and $b_n = n^p$ in the result of Question 4 shows that the series converges.

Case 2: $p \geq 0$.

$n^p(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}) \geq \frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}$. We will show that $\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}} \geq \frac{1}{n}$ for large n. This is same as showing $\sqrt{n}(\sqrt{\frac{n}{n-1}} - 1) \geq 1$ for large n. By squaring both sides this is equivalent to showing $n(2n - 1 - 2n\sqrt{\frac{1}{1-1/n}} + 2\sqrt{\frac{1}{1-1/n}}) \geq 1$ whiich easily seen to be true for all large n. □